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# Internally resonant surface waves in a circular cylinder

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The two dominant, linearly independent surface-wave modes in a circular cylinder, which differ only by an azimuthal rotation of $\frac{1}{2}\pi$ and have equal natural frequencies, are nonlinearly coupled, both directly and through secondary modes. The corresponding, weakly nonlinear free oscillations are described by a pair of slowly modulated sinusoids, the amplitudes and phases of which are governed by a four-dimensional Hamiltonian system that is integrable by virtue of conservation of energy and angular momentum. The resulting solutions are harmonic in a particular, slowly rotating reference frame. Harmonic oscillations in the laboratory reference frame are realized for three special sets of initial conditions and correspond to a standing wave with a fixed nodal diameter and to two azimuthally rotating waves with opposite senses of rotation. The finite-amplitude corrections to the natural frequencies of these harmonic oscillations are calculated as functions of the aspect ratio $d/a$ (depth/radius). A small neighbourhood of $d/a = 0.1523$, in which the natural frequency of the dominant axisymmetric mode approximates twice that of the two dominant antisymmetric modes, is excluded. Weak, linear damping is incorporated through a transformation that renders the evolution equations for the damped system isomorphic to those for the undamped system.

## 1. Introduction

I consider here weakly nonlinear free oscillations of the dominant free-surface modes in a circular cylinder of radius $a$ and ambient depth $d$, starting from the Lagrangian formulation developed in an earlier paper (Miles 1976); equations and sections from that paper are cited by the prefix I. These dominant modes differ only by an azimuthal rotation of $\frac{1}{2}\pi$ and have the same natural frequency $\omega_1$. They are uncoupled (orthogonal) in the linear approximation, but are nonlinearly coupled both through direct, third-order interactions and through secondary modes that are excited at second order and effect indirect, third-order interactions. The resulting internal resonance resembles that for a spherical pendulum, but is more complicated in consequence of the participation of the secondary modes.

A more common internal resonance is between two orthogonal modes with natural frequencies in the approximate ratio 2:1. The direct coupling is then quadratic, the indirect coupling through secondary modes is of higher order and therefore negligible, and the solution may be obtained through a Hamiltonian formulation with a canonical representation of the slowly varying amplitudes and phases (I, §7). I posit a similar representation for the primary modes in the present problem, but, in order to incorporate the secondary modes, find it expedient to start (in §2) from the Lagrangian of the system, expressed as a function of the generalized coordinates for the complete set of normal modes. I then (in §3) average this Lagrangian over the natural period of the primary oscillation and eliminate the amplitudes of the secondary modes (by invoking Hamilton's principle for the average Lagrangian) to obtain a reduced Lagrangian that depends only on the slowly varying amplitudes and phases of the dominant modes. The resulting four-dimensional Hamiltonian system admits two constants of the motion (which are measures of the energy and angular momentum in the original system), by virtue of which the system is integrable.

I obtain the general solution of this Hamiltonian system in §4 through a transformation to a slowly rotating reference frame in which the motion is harmonic with the angular frequency $\omega$ given by

$$\left(\frac{\omega}{\omega_1}\right)^2 = 1 - A\frac{\overline{\eta^2}}{\lambda^2}, \quad \lambda = k^{-1}\tanh kd, \tag{1.1 a, b}$$

where $\overline{\eta^2}$ is the mean-square (averaged over both space and time) elevation of the surface wave, $\lambda$ is a reference length that reduces to $1/k$ for deep water and to $d$ for shallow water, $k = 1.8412/a$ is the wavenumber of the dominant mode, and $A$, which depends only on the aspect ratio $d/a$ of the cylinder, is plotted in figure 1. Harmonic motion in the laboratory reference frame is realized only for each of three special sets of initial conditions, one of which yields a standing wave with a fixed nodal diameter and the other two of which yield azimuthally rotating waves with opposite senses of rotation. The angular frequency $\omega$ of the standing wave is given by (1.1); that of the rotating waves is given by a similar result, with $A$ replaced by $A+B$ (see figure 2).

The result (1.1 $a$) is a special case of I (6.5), which gives the finite-amplitude correction to the natural frequency of any mode in any cylinder for which the normal modes are known and resonant coupling is absent. The corresponding result for two-dimensional standing waves in deep water is obtained by setting $A = 1$ (Rayleigh 1915). The corresponding result for the lowest axisymmetric mode (one nodal circle) in a circular cylinder has been obtained by Mack (1962), whilst that for the simplest three-dimensional mode in a rectangular cylinder has been obtained by Verma & Keller (1962).

Weak, linear damping may be incorporated in the formulation of §§3 and 4 simply by adding the appropriate terms to the equations of motion (§6). The system remains integrable after a transformation that renders the evolution equations for the damped system isomorphic to those for the undamped system.

It is implicit in the following analysis that none of the secondary natural frequencies approximates $2\omega_1$, which would imply resonant coupling with the primary modes. The critical condition is

$$4\omega_1^2/g = 4k\tanh kd = k_{ij}\tanh k_{ij}d, \tag{1.2}$$

where $k_{ij}$ is the wavenumber of the secondary mode (with $i$ nodal diameters and $j-1+\delta_{i0}$ nodal circles for $i = 0, 1, \ldots$ and $j = 1, 2, \ldots$), and $k \equiv k_{11}$. Only the $0j$ and $2j$ modes are excited by the primary 11 modes, and (1.2) can be satisfied only for $k_{01}a = 3.8317$, $k_{02}a = 7.0156$ and $k_{22}a = 6.7061$, for which the critical depths are given by $d/a = 0.1523$, 1.010 and 0.8314, respectively. The relevant coupling coefficient (see (A 11)) proved to be numerically small (the dimensionless bandwidths of the resonances are of the order of $10^{-4}$) in the latter two cases, in consequence of which damping is likely to vitiate the resonances, but the first resonance, which implies $|A|, |B| > 10$ in $0.12 \leqslant d/a \leqslant 0.18$ (note that this resonance does not affect

$A+B$), is manifestly significant. The proper analysis of this resonance, in which the dominant axisymmetric 01 mode would have to be treated on the same level as the two dominant antisymmetric 11 modes, would lead to a six-dimensional Hamiltonian system, which presumably would not be (exactly) integrable and therefore might admit chaotic solutions. Absent this analysis, it must be presumed that the present results are invalid in some neighbourhood of $d/a = 0.1523$.

*Two-degree-of-freedom model*

The calculation of the Lagrangian and its average in §§2 and 3 below is necessarily rather complicated; accordingly, on the recommendation of one of the referees, I first present the corresponding formulation for a two-degree-of-freedom model.

Consider two identical oscillators of generalized mass $m$, resonant frequency $\omega$ and generalized coordinates $\eta_1$ and $\eta_2$ (which may be assumed to be dimensionless) with the Lagrangian

$$L = \tfrac{1}{2}m\{\dot\eta_1^2+\dot\eta_2^2-\omega^2[(\eta_1^2+\eta_2^2)+\gamma(\eta_1^2+\eta_2^2)^2]\}, \tag{1.3}$$

where $\gamma$ is a measure of the nonlinearity. The system is linear for $\gamma = 0$ and then admits harmonic solutions of the form $\eta_n = A_n \cos \omega t + B_n \sin \omega t$. The corresponding solution for weak nonlinearity ($\gamma > 0$ and $\eta_1^2+\eta_2^2 \ll 1$) is a slowly modulated sinusoid, which may be posed in the (Van der Pol) form

$$\eta_n = \epsilon\{p_n(\tau) \cos \omega t + q_n(\tau) \sin \omega t\} \quad (n = 1, 2), \tag{1.4}$$

where $p_n$ and $q_n$ are slowly varying amplitudes, and $\tau = \tfrac{1}{2}\epsilon^2\omega t$ is a slow time (the scaling is chosen to simplify the subsequent development). The average Lagrangian, as obtained through the substitution of (1.4) into (1.3) followed by averaging over a $2\pi$ interval of $\omega t$, is then given by

$$\langle L\rangle = \tfrac{1}{2}\epsilon^4 m\omega^2\{\tfrac{1}{2}(\dot p_n q_n - p_n \dot q_n) + H\}, \tag{1.5}$$

where the dots now signify differentiation with respect to $\tau$, the repeated index $n$ is summed over 1 and 2, and

$$H = \tfrac{1}{2}\gamma\{-\tfrac{3}{4}(p_1^2+q_1^2+p_2^2+q_2^2)^2+(p_1 q_2 - p_2 q_1)^2\}. \tag{1.6}$$

The requirement that $\langle L\rangle$ be stationary with respect to variations of each of the $p_n$ and $q_n$ yields the canonical equations

$$\dot p_n = -\frac{\partial H}{\partial q_n}, \quad \dot q_n = \frac{\partial H}{\partial p_n}, \tag{1.7$a, b$}$$

in which $H$ appears as a Hamiltonian and therefore is a constant of the motion. The functionals $E \equiv \tfrac{1}{2}(p_1^2+q_1^2+p_2^2+q_2^2)$ and $M \equiv p_1 q_2 - p_2 q_1$, which are measures of energy and (in a generalized sense) angular momentum, also may be shown (through the calculation of $\dot E$ and $\dot M$) to be constants of the motion, by virtue of which (1.7) may be integrated exactly (as in §4). Damping may be incorporated as in §6.

## 2. The Lagrangian

We pose the free-surface displacement in the form

$$\eta(r, \theta, t) = \eta_n(t)\,\psi_n(r, \theta), \tag{2.1}$$

where $r$ and $\theta$ are plane polar coordinates, the repeated indices are summed over $1, \ldots, N$, and the $\eta_n$ are generalized coordinates; the $\psi_n$ are members of the complete set of orthogonal eigenfunctions determined by

$$(\nabla^2+k_n^2)\psi_n = 0, \quad \frac{\partial\psi_n}{\partial r} = 0 \quad (r=a), \quad [\psi_m\psi_n] = \delta_{mn}, \qquad (2.2a,b,c)$$

where $k_n$ is the eigenvalue, the square brackets signify a spatial average,

$$[f(r,\theta)] \equiv (\pi a^2)^{-1}\int_0^a\int_0^{2\pi} fr\,\mathrm{d}r\,\mathrm{d}\theta, \qquad (2.3)$$

and $\delta_{mn}$ is the Kronecker delta. Each of the eigenfunctions for the circular basin requires three indices for its complete specification (the azimuthal wavenumber, the radial wavenumber, and, except for the axisymmetric modes, the superscript c or s to distinguish between cosine and sine azimuthal variations), and the use of a single index in (2.1) and subsequently is merely a convenient abbreviation. The required results are

$$\psi_{mn}^{\mathrm{c,s}} = N_{mn}^{-1}J_m(k_{mn}r)(\cos m\theta, \sin m\theta) \quad (m=0,1,\ldots, \quad n=1,2,\ldots), \quad (2.4a)$$

$$J'_m(k_{mn}a) = 0, \quad N_{mn}^2 = \tfrac{1}{2}(1+\delta_{0m})\left\{1-\left(\frac{m}{k_{mn}a}\right)^2\right\}J_m^2(k_{mn}a), \qquad (2.4b,c)$$

where $J_m$ is a Bessel function and $k_{mn}$ is one of the infinite, discrete set of eigenvalues determined by (2.4*b*).† We reserve the single subscripts 1 and 2 for the dominant (or primary) modes according to

$$\psi_{1,2} \equiv \psi_{11}^{\mathrm{c,s}} = N^{-1}J_1(kr)(\cos\theta, \sin\theta) \quad (N\equiv N_{11}, \quad k\equiv k_{11} = 1.8412/a). \quad (2.5)$$

The kinetic energy corresponding to (2.1) is given by I (3.1) in the form

$$T = \tfrac{1}{2}\rho S a_{mn}\dot\eta_m\dot\eta_n \quad (\dot\eta_n \equiv \mathrm{d}\eta_n/\mathrm{d}t), \qquad (2.6)$$

where the inertial coefficients $a_{mn}$, which have the dimensions of length, depend on the vector $\{\eta_n\}$, and, in the present investigation, may be approximated by the quadratic truncation I (3.3):

$$a_{mn} = \delta_{mn}a_m + a_{lmn}\eta_l + \tfrac{1}{2}a_{jlmn}\eta_j\eta_l, \qquad (2.7)$$

where

$$a_n = k_n^{-1}\coth k_n d \equiv g/\omega_n^2 \qquad (2.8)$$

and $\omega_n$ are the length of an equivalent pendulum and the natural frequency of the $n$th mode,

$$a_{lmn} = C_{lmn} - D_{lmn}a_m a_n, \qquad (2.9a)$$

$$a_{jlmn} = -D_{jlmn}(a_m+a_n) + 2D_{jmi}D_{lni}a_i a_m a_n, \qquad (2.9b)$$

and

$$C_{lmn} = [\psi_l\psi_m\psi_n], \qquad (2.10a)$$

$$D_{lmn} = [\psi_l\nabla\psi_m\cdot\nabla\psi_n], \quad D_{jlmn} = [\psi_j\psi_l\nabla\psi_m\cdot\nabla\psi_n]. \qquad (2.10b,c)$$

The required correlation integrals defined by (2.10) are evaluated in Appendix A, where it is found that the contributions of $\psi_{01}$, $\psi_{21}^{\mathrm{c}}$ and $\psi_{21}^{\mathrm{s}}$ to the Lagrangian dominate those of the remaining secondary modes by at least an order of magnitude.

† The eigenfunction $\psi_{00} = 1$ ($k_{00} = 0$) is excluded from $\eta$, and hence from the potential energy, by conservation of mass, which requires $[\eta] = [\eta_{00}] = 0$. It obviously makes no contribution to the kinetic energy. It does, however, contribute to the velocity potential and enters the calculation of the pressure (I, §5).

The potential energy corresponding to (2.1) is given by

$$V = \tfrac{1}{2}\rho g \int_0^a \int_0^{2\pi} \eta^2 r \,\mathrm{d}r\,\mathrm{d}\theta = \tfrac{1}{2}\rho g S \eta_n \eta_n. \tag{2.11}$$

Combining (2.6)–(2.8) and (2.11), we obtain the truncated Lagrangian (cf. (1.3))

$$L \equiv (\rho S)^{-1}(T-V) = \tfrac{1}{2}a_n(\dot{\eta}_n^2 - \omega_n^2 \eta_n^2) + \tfrac{1}{2}a_{lmn}\eta_l \dot{\eta}_m \dot{\eta}_n + \tfrac{1}{4}a_{jlmn}\eta_j \eta_l \dot{\eta}_m \dot{\eta}_n. \tag{2.12}$$

## 3. The average Lagrangian

We posit the dominant modes in the form (cf. (1.4))

$$\eta_n = \epsilon\lambda\{p_n(\tau)\cos\omega_1 t + q_n(\tau)\sin\omega_1 t\} \quad (n = 1, 2), \tag{3.1}$$

where $\epsilon$ is a small parameter (which ultimately will be reabsorbed in the solution), $\lambda$ is the reference length (1.1*b*), $p_n$ and $q_n$ are slowly varying, dimensionless amplitudes, and

$$\tau = \tfrac{1}{2}\epsilon^2 \omega_1 t \tag{3.2}$$

is a dimensionless, slow time. The remaining (*secondary*) modes are forced by terms in the equations of motion that are quadratic in $\eta_1$ and $\eta_2$, in consequence of which they must be $O(\epsilon^2)$ and have carrier frequencies 0 and $2\omega_1$; accordingly, we posit

$$\eta_n = \epsilon^2\lambda\{A_n(\tau)\cos 2\omega_1 t + B_n(\tau)\sin 2\omega_1 t + C_n(\tau)\} \quad (n \neq 1, 2). \tag{3.3}$$

It also follows from scaling considerations that two of the terms in the triple product $\eta_l \dot{\eta}_m \dot{\eta}_n$ in $L$ must correspond to the dominant modes, and $a_{lmn}$ then differs from zero only if the third term corresponds to a secondary mode with azimuthal wavenumber 0 or 2 (the products of the dominant modes introduce $\sin^2\theta$, $\cos^2\theta$ and $\sin 2\theta$ in the correlation integrals from which $a_{lmn}$ is derived, and these products are orthogonal to $\cos m\theta$ and $\sin m\theta$ unless $m = 0$ or $2$); accordingly, we may write (note that $a_{nlm} = a_{nml}$)

$$a_{lmn}\eta_l \dot{\eta}_m \dot{\eta}_n = a_{lmn}\eta_l \dot{\eta}_m \dot{\eta}_n + a_{mln}\eta_m \dot{\eta}_l \dot{\eta}_n + a_{nml}\eta_n \dot{\eta}_m \dot{\eta}_l \tag{3.4a}$$

$$= a_{lmn}\eta_l \dot{\eta}_m \dot{\eta}_n + 2a_{nml}\eta_n \dot{\eta}_m \dot{\eta}_l, \tag{3.4b}$$

on the right-hand sides of which $m$ and $n$ are summed only over 1, 2, and $l$ is summed only over the secondary modes.

Only the dominant modes contribute to the quartic term in $L$ in the present approximation. Inferring from (2.5), (2.9*b*) and (2.10*b, c*) that

$$a_{2222} = a_{1111}, \quad a_{2211} = a_{1122}, \quad a_{2121} = a_{1212}, \quad a_{2112} = a_{1221}, \tag{3.5}$$

and that the remaining $a_{jlmn}$ $(j, l, m, n = 1 \text{ or } 2)$ vanish, we obtain

$$a_{jlmn}\eta_j \eta_l \dot{\eta}_m \dot{\eta}_n$$
$$= a_{1111}(\eta_1^2 \dot{\eta}_1^2 + \eta_2^2 \dot{\eta}_2^2) + a_{1122}(\eta_1^2 \dot{\eta}_2^2 + \eta_2^2 \dot{\eta}_1^2) + 2(a_{1212} + a_{1221})\eta_1 \eta_2 \dot{\eta}_1 \dot{\eta}_2. \tag{3.6}$$

Substituting (3.1), (3.3), (3.4) and (3.6) into (2.12) and averaging $L$ over a $2\pi$

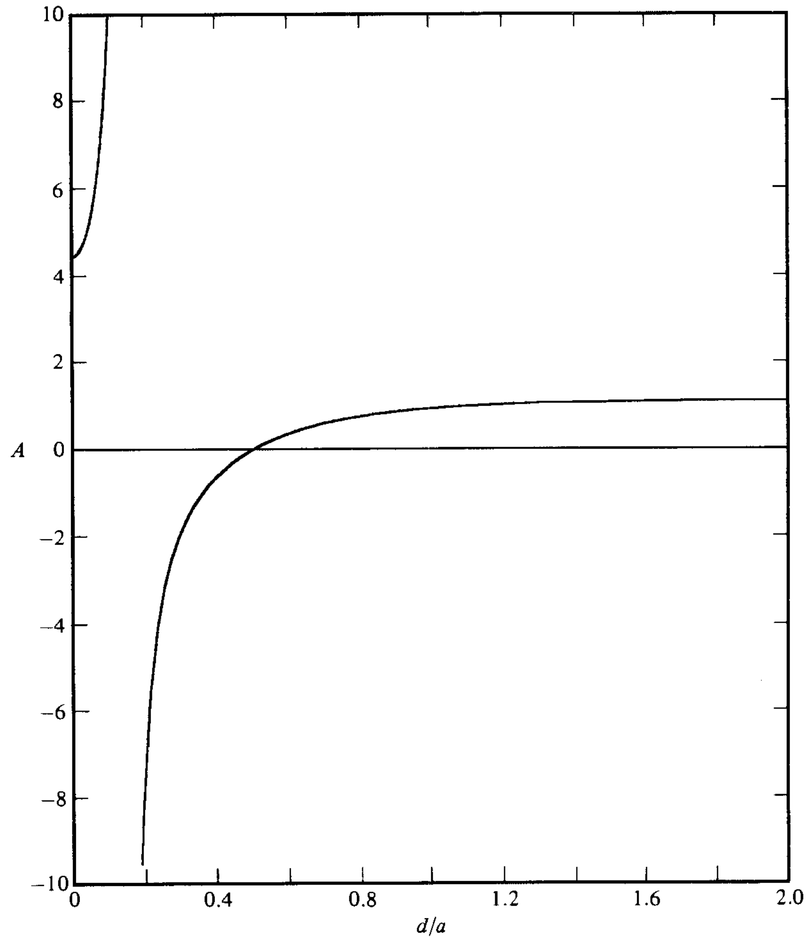


FIGURE 1. The parameter $A$, as determined in Appendix A, versus depth/radius for the circular cylinder. $A = (4.430, \infty, 0, 1.112)$ at $d/a = (0, 0.1523, 0.5059, \infty)$. See figure 3 for expanded plot.

interval of $\omega_1 t$ while holding $\tau$ fixed (this average is denoted by $\langle\ \rangle$), we obtain

$$\begin{aligned}\langle L\rangle = \tfrac{1}{2}\epsilon^4 g\lambda^2\Big[&\tfrac{1}{2}(\dot{p}_n q_n - p_n \dot{q}_n) + \frac{1}{4}\frac{\lambda^2}{a_1}\{a_{1111}(E_1^2+E_2^2) \\ &+ 2a_{1122}(E_1 E_2 + \tfrac{1}{2}M^2) + 2(a_{1212}+a_{1221})(E_1 E_2 - \tfrac{1}{2}M^2)\} \\ &+ \frac{1}{2}\frac{\lambda}{a_1}\{(2a_{nml} - \tfrac{1}{2}a_{lmn})(A_l(p_m p_n - q_m q_n) + B_l(p_m q_n + p_n q_m)) \\ &+ a_{lmn} C_l(p_m p_n + q_m q_n)\} + \tfrac{1}{2}\ell_l(A_l^2+B_l^2) - C_l C_l\Big],\end{aligned} \tag{3.7}$$

wherein the summations are restricted as on the right-hand side of (3.4), an error factor of $1+O(\epsilon^2)$ is implicit, the dots now signify differentiation with respect to $\tau$,

$$E_n = \tfrac{1}{2}(p_n^2+q_n^2), \quad E = E_1+E_2, \quad M = p_1 q_2 - p_2 q_1 \tag{3.8a, b}$$

are measures of the energy and angular momentum (see Appendix B) in the dominant modes, and

$$\ell_l = \frac{4\omega_1^2-\omega_l^2}{\omega_l^2} = \frac{4a_l}{a_1} - 1. \tag{3.9}$$

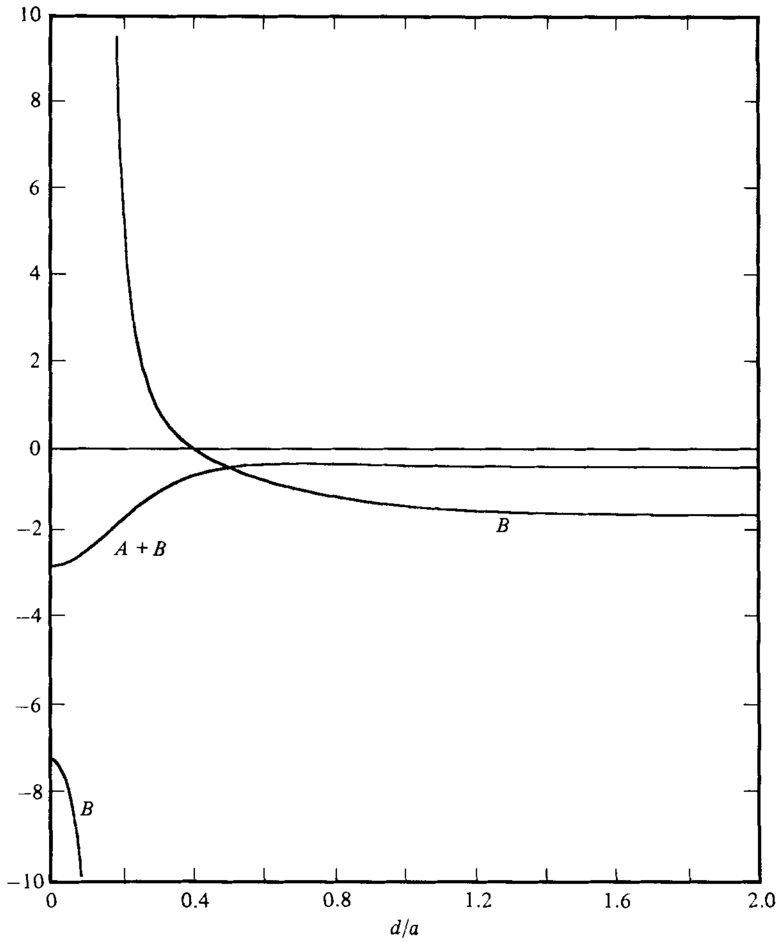


FIGURE 2. The parameters $B$ and $A+B$, as determined in Appendix A, versus depth/radius for the circular cylinder. $B = (-7.147, \infty, 0, 1.531)$ at $d/a = (0, 0.1523, 0.4063, \infty)$. See figure 3 for expanded plot.

It is implicit that $\mathscr{E}_l \gg \epsilon^2$. This condition may be violated for the 01, 02 or 21 modes if $d/a$ approximates 0.1523, 1.010 or 0.8314 respectively (see last paragraph in §1).

Requiring $\langle L \rangle$ to be stationary with respect to each of $A_l$, $B_l$ and $C_l$, we obtain

$$\{A_l, B_l\} = -\frac{1}{2}\frac{\lambda}{\alpha_1 \mathscr{E}_l}(2\alpha_{nml} - \tfrac{1}{2}\alpha_{lmn})\{p_m p_n - q_m q_n, p_m q_n + p_n q_m\} \tag{3.10a}$$

and

$$C_l = \frac{1}{4}\frac{\lambda}{\alpha_1}\alpha_{lmn}(p_m p_n + q_m q_n) \tag{3.10b}$$

(note that $\lambda/\alpha_1 = \tanh^2 kd$). The substitution of (3.10) into (3.7) yields $\langle L \rangle$ as a function of the four variables $p_1, q_1, p_2, q_2$. Invoking (A 10) and (A 21) of Appendix A, we place the result in the form (cf. 1.5))

$$\langle L \rangle = \tfrac{1}{2}\epsilon^4 g \lambda^2 \{\tfrac{1}{2}(\dot{p}_n q_n - p_n \dot{q}_n) + H\}, \tag{3.11}$$

where (cf. (1.6))

$$H = \tfrac{1}{2}AE^2 + \tfrac{1}{2}BM^2 \tag{3.12a}$$

$$= \tfrac{1}{8}A(p_1^2 + q_1^2 + p_2^2 + q_2^2)^2 + \tfrac{1}{2}B(p_1 q_2 - p_2 q_1)^2, \tag{3.12b}$$

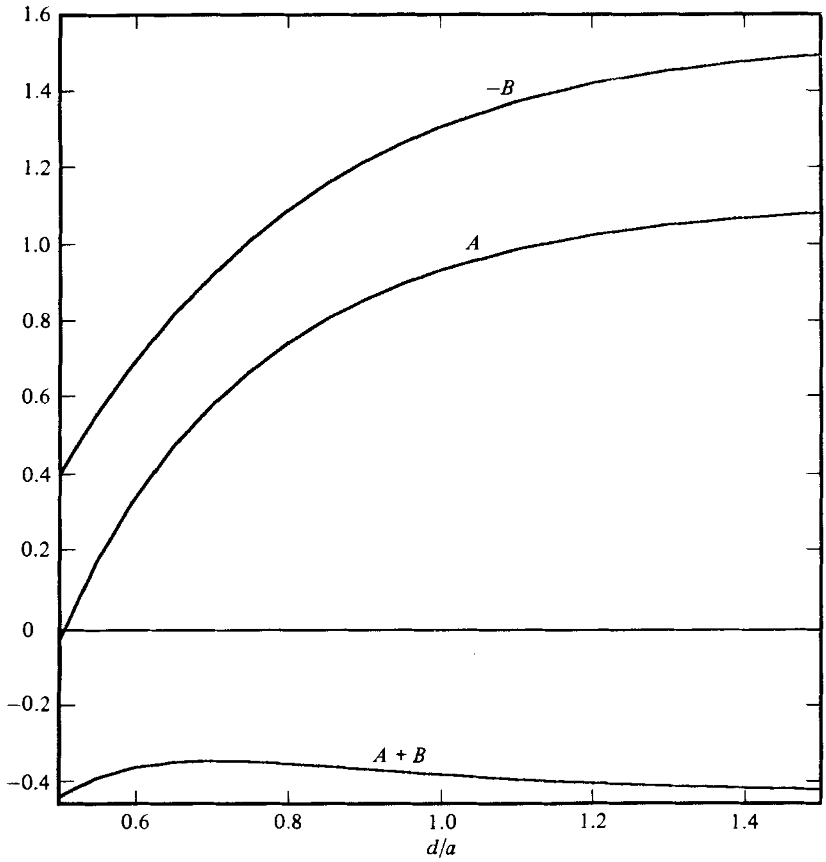


FIGURE 3. Expanded plots of $A$, $-B$ and $A+B$ in the domain of positive $A$.

and the parameters $A$ and $B$ (which depend only on the aspect ratio $d/a$) are given by (A 23) and are plotted in figures 1 and 2.

## 4. Evolution equations

Requiring $\langle L\rangle$ (3.11) to be stationary with respect to each of the $p_n$ and $q_n$, we obtain the canonical equations

$$\dot{p}_n = -\frac{\partial H}{\partial q_n}, \quad \dot{q}_n = \frac{\partial H}{\partial p_n} \quad (n = 1, 2), \tag{4.1\,a, b}$$

in which $H$ appears as a Hamiltonian, and therefore (since it does not contain $\tau$ explicitly) is a constant of the motion. Substituting $H$ from (3.12) into (4.1) and introducing

$$r_n = p_n + \mathrm{i}q_n, \quad \boldsymbol{r} = (r_1, r_2), \tag{4.2\,a, b}$$

we obtain

$$\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}\tau} = \begin{bmatrix} \mathrm{i}AE & BM \\ -BM & \mathrm{i}AE \end{bmatrix} \boldsymbol{r}. \tag{4.3}$$

It may be inferred from (4.1), or otherwise directly from conservation of energy and angular momentum, that $E$ and $M$ are constants of the motion; accordingly (4.3) is linear and admits the general solution

$$\boldsymbol{r}(\tau) = \boldsymbol{R}(BM\tau)\,\boldsymbol{r}_0\,\mathrm{e}^{\mathrm{i}AE\tau}, \tag{4.4}$$

where the matrix

$$\boldsymbol{R}(\phi) = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}, \tag{4.5}$$

*qua* linear operator, rotates its vector operand through the angle $\phi$, and $\boldsymbol{r}_0$ is the initial value of $\boldsymbol{r}$.

## 5. Free-surface motion

The free-surface displacement (2.1) may be approximated by

$$\eta_{\mathrm{d}} = \eta_1\psi_1 + \eta_2\psi_2 \tag{5.1a}$$

$$= N^{-1}J_1(kr)\,(\eta_1\cos\theta + \eta_2\sin\theta) \tag{5.1b}$$

within $1+O(\epsilon)$, where the subscript d identifies the contribution of the dominant modes, and (5.1*b*) follows from (5.1*a*) through (2.5). Combining (3.1), (4.2) and (4.4), we obtain

$$(\eta_1, \eta_2) = \boldsymbol{R}(-\Omega t)\,(\eta_1', \eta_2'), \tag{5.2}$$

where

$$\eta_n' = \epsilon\lambda(p_{n0}\cos\omega t + q_{n0}\sin\omega t) \quad (n = 1, 2), \tag{5.3}$$

$$\frac{\omega}{\omega_1} = 1 - \tfrac{1}{2}\epsilon^2 AE, \quad \frac{\Omega}{\omega_1} = -\tfrac{1}{2}\epsilon^2 BM. \tag{5.4a, b}$$

Substituting (5.2) into (5.1*b*) and invoking (4.5), we obtain

$$\eta_{\mathrm{d}} = N^{-1}J_1(kr)\,(\eta_1'\cos\theta' + \eta_2'\sin\theta'), \tag{5.5}$$

where

$$\theta' = \theta - \Omega t. \tag{5.6}$$

It follows from (5.5) that $\eta_{\mathrm{d}}$ comprises a pair of orthogonal, simple harmonic, standing waves of frequency $\omega$ in a reference frame that rotates with the angular velocity $\Omega$. In the laboratory reference frame, $\eta_{\mathrm{d}}$ is quasi-periodic and comprises the two frequencies $\omega \pm \Omega$. There are, however, three special cases in which the motion is harmonic in the laboratory reference frame.

The first special case is $M = 0$, for which $\eta_{\mathrm{d}}$ reduces to a standing wave with a fixed nodal diameter. Choosing $\theta = \frac{1}{2}\pi$ at this nodal diameter, so that $\eta_2 \equiv 0$, and measuring $t$ from the time of maximum displacement, so that $q_{10} \equiv 0$, we obtain

$$\eta_{\mathrm{d}} = \eta_1\psi_1 = (2\overline{\eta^2})^{\frac{1}{2}} N^{-1}J_1(kr)\cos\theta\cos\omega t \quad (M = 0), \tag{5.7}$$

where

$$\overline{\eta^2} \equiv \langle[\eta_{\mathrm{d}}^2]\rangle = \epsilon^2\lambda^2 E \tag{5.8}$$

(note that $\langle[\eta_{\mathrm{d}}^2]\rangle = \langle[\eta^2]\rangle$ within $1+O(\epsilon^2)$). Eliminating $\epsilon^2$ between (5.4*a*) and (5.8), we obtain

$$\frac{\omega}{\omega_1} = 1 - \tfrac{1}{2}A\frac{\overline{\eta^2}}{\lambda^2}, \tag{5.9}$$

which is equivalent to (1.1*a*) within the present approximation.

The remaining special cases are $M = \pm E$. Remarking that, from (3.8),

$$2(E \mp M) = (p_1 \mp q_2)^2 + (p_2 \pm q_1)^2, \tag{5.10}$$

where, here and subsequently, the alternative signs are vertically ordered, we infer that $M = \pm E$ implies $p_2 = \mp q_1$ and $q_2 = \pm p_1$ or, equivalently, $r_2 = \pm i r_1$. Measuring $t$ such that $q_{10} \equiv 0$, we then may reduce (5.1$b$) to

$$\eta_d = (\overline{\eta^2})^{\frac{1}{2}} (N^{-1} J_1(kr) \cos(\omega t \mp \theta) \quad (M = \pm E), \tag{5.11}$$

wherein $\overline{\eta^2}$ is given by (5.8), the alternative signs are vertically ordered, and

$$\frac{\omega}{\omega_1} = 1 - \tfrac{1}{2}(A+B)\frac{\overline{\eta^2}}{\lambda^2}. \tag{5.12}$$

## 6. Damped motion

Linear damping may be incorporated in the preceding formulation by adding $\alpha(p_n, q_n)$ to the left-hand sides of (4.1$a, b$) and $\alpha \boldsymbol{r}$ to the left-hand side of (4.3) (see (6.5) below), where

$$\alpha = 2\delta/\epsilon^2, \tag{6.1}$$

and $\delta$ is the damping ratio ($2\pi\delta$ is the logarithmic decrement) of free oscillations in the dominant mode.† (Note that damping dominates the nonlinear effects considered here if $\delta \gg \epsilon^2$.) $E$ and $M$ are no longer constants, but it follows from the modified evolution equations that they satisfy

$$\dot{E} + 2\alpha E = 0, \quad \dot{M} + 2\alpha M = 0, \tag{6.2$a, b$}$$

and hence that

$$E = E_0 \mathrm{e}^{-2\alpha\tau}, \quad M = M_0 \mathrm{e}^{-2\alpha\tau}, \tag{6.3$a, b$}$$

where $E_0$ and $M_0$ are initial values. This suggests the change of variables

$$\boldsymbol{r} = \hat{\boldsymbol{r}} \mathrm{e}^{-\alpha\tau}, \quad \hat{\tau} = (2\alpha)^{-1}(1 - \mathrm{e}^{-2\alpha\tau}), \tag{6.4$a, b$}$$

under which

$$\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}\tau} + \alpha \boldsymbol{r} = \begin{bmatrix} \mathrm{i}AE & BM \\ -BM & \mathrm{i}AE \end{bmatrix} \boldsymbol{r} \tag{6.5}$$

transforms to

$$\frac{\mathrm{d}\hat{\boldsymbol{r}}}{\mathrm{d}\hat{\tau}} = \begin{bmatrix} \mathrm{i}AE_0 & BM_0 \\ -BM_0 & \mathrm{i}AE_0 \end{bmatrix} \hat{\boldsymbol{r}}, \tag{6.6}$$

which is isomorphic to (4.3). It then follows from (4.4) and (6.4) that

$$\boldsymbol{r}(\tau) = \boldsymbol{R}(BM_0 \hat{\tau}) \boldsymbol{r}_0 \exp(-\alpha\tau + \mathrm{i}AE_0\hat{\tau}). \tag{6.7}$$

Proceeding as in §5, we obtain (note that $\alpha\tau = \delta\omega_1 t$)

$$\eta_d = N^{-1} J_1(kr)(\hat{\eta}_1 \cos\hat{\theta} + \hat{\eta}_2 \sin\hat{\theta}) \mathrm{e}^{-\delta\omega_1 t}, \tag{6.8}$$

where

$$\hat{\eta}_n = \epsilon\lambda\{p_{n0} \cos(\omega_1 t - AE_0\hat{\tau}) + q_{n0} \sin(\omega_1 t - AE_0 \hat{\tau})\} \quad (n = 1, 2), \tag{6.9}$$

$$\hat{\theta} = \theta + BM_0\hat{\tau}. \tag{6.10}$$

The frequency in, and the angular velocity of, the rotating reference frame, as

† The logarithmic decrement may be obtained either by direct measurement of the decay of the dominant mode in the form (5.7) or through semiempirical calculation (Miles 1967).

obtained by differentiating the phase of $\hat{\eta}$ and $-\hat{\theta}$ with respect to $t$, are given by

$$\frac{\omega}{\omega_1} = 1-\tfrac{1}{2}\epsilon^2 A E_0 \,\mathrm{e}^{-2\delta\omega_1 t}, \quad \frac{\Omega}{\omega_1} = -\tfrac{1}{2}\epsilon^2 B M_0 \,\mathrm{e}^{-2\delta\omega_1 t}, \tag{6.11 a, b}$$

which also are obtained by substituting (6.3) into (5.4).

The damped counterparts of (5.7) and (5.11) may be obtained by choosing $M_0 = q_{10} = p_{20} = q_{20} = 0$ or $M_0 = \pm E_0$, $q_{20} = \pm p_{10}$, $q_{10} = p_{20} = 0$.

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## Appendix A. Correlation integrals and inertial coefficients

The correlation integrals $C_{lmn}$ (2.10$a$) and $D_{lmn}$ (2.10$b$) are related by I (2.18),

$$D_{lmn} = \tfrac{1}{2}(k_m^2 + k_n^2 - k_l^2)\, C_{lmn}, \tag{A 1}$$ †

the substitution of which into (2.9$a$) yields

$$\alpha_{lmn} = C_{lmn}\{1 + \tfrac{1}{2}(k_l^2 - k_m^2 - k_n^2)\, \alpha_m \alpha_n\}. \tag{A 2}$$

As noted in §3, two of the three indices of $\alpha_{lmn}$ must correspond to the dominant modes (for which $k_2 = k_1 \equiv k$ and $\alpha_2 = \alpha_1$), whilst the third must correspond to a mode with azimuthal wavenumber 0 or 2. For the calculation of $\langle L \rangle$ (3.7), we require

$$\alpha_{lmn} = C_{lmn}\{1 + (\tfrac{1}{2}k_l^2 - k^2)\, \alpha_1^2\} \quad (m, n = 1 \text{ or } 2) \tag{A 3}$$

and
$$2\alpha_{nml} - \tfrac{1}{2}\alpha_{lmn} = C_{lmn}\{\tfrac{3}{2} + (\tfrac{1}{2}k^2 - \tfrac{1}{4}k_l^2)\,\alpha_1^2 - k_l^2\, \alpha_1 \alpha_l\} \quad (m, n = 1 \text{ or } 2). \tag{A 4}$$

Substituting (2.4$a$) and (2.5$a$) into (2.10$a$) and expanding the index prescriptions as in (2.4), we obtain (note that $C_{lmn}$ is invariant under every permutation of its three indices)

$$C_{11i} \equiv C_{11ij}^{\mathrm{c,\,s}} = [\psi_1^2\, \psi_{ij}^{\mathrm{c,\,s}}] \tag{A 5a}$$

$$= (\pi a^2 N^2 N_{ij})^{-1} \int_0^a \int_0^{2\pi} J_1^2(kr)\, J_i(k_{ij} r) \cos^2\theta\, (\cos i\theta, \sin i\theta)\, r\, \mathrm{d}r\, \mathrm{d}\theta, \tag{A 5b}$$

$$= \delta_{0i} I_{0j} + \delta_{2i} I_{2j},\ 0 \tag{A 5c}$$

where, here and subsequently, the c, s alternatives are horizontally ordered, and

$$I_{ij} = (1+\delta_{0i})\,(2k^2 a^2 N^2 N_{ij})^{-1} \int_0^{ka} J_1^2(x)\, J_i(k_{ij} x/k)\, x\, \mathrm{d}x. \tag{A 6}$$

Similarly,
$$C_{22ij}^{\mathrm{c,\,s}} = \delta_{0i} I_{0j} - \delta_{2i} I_{2j},\ 0, \tag{A 7}$$

$$C_{12ij}^{\mathrm{c,\,s}} = 0,\ \delta_{2i} I_{2j}. \tag{A 8}$$

The $I_{ij}$ integrals are displayed in table 1, from which it is evident that $\psi_{01}$, $\psi_{21}^{\mathrm{c}}$ and $\psi_{21}^{\mathrm{s}}$ dominate the remaining secondary modes.

We now use the preceding results to simplify the terms in $A_l$, $B_l$ and $C_l$ in (3.7).

† Repeated indices are *not* summed in this Appendix except as explicitly indicated.

| $j$ | $i = 0$ | $i = 2$ |
|---|---|---|
| 1 | 0.40994 | 0.7643 |
| 2 | $6.470 \times 10^{-3}$ | $-6.354 \times 10^{-3}$ |
| 3 | $1.232 \times 10^{-3}$ | $-1.087 \times 10^{-3}$ |
| 4 | $3.93 \times 10^{-4}$ | $-3.56 \times 10^{-4}$ |

TABLE 1. The integrals $I_{ij}$ (A 6)

Expanding the index prescription in (3.10), summing over $m$, $n = 1, 2$, and invoking (A 3), (A 4), (A 5$c$), (A 7) and (A 8), we obtain

$$A_l \equiv A_{ij}^{c,s} = -\frac{\lambda}{a_1}\mathscr{E}_l^{-1}(\tfrac{3}{2}+\tfrac{1}{2}k^2 a_1^2 - \tfrac{1}{4}k_{ij}^2 a_1^2 - k_{ij}^2 a_1 a_{ij}) I_{ij}$$
$$\times \{\delta_{0i}(p_1^2 - q_1^2 + p_2^2 - q_2^2) + \tfrac{1}{2}\delta_{2i}(p_1^2 - q_1^2 - p_2^2 + q_2^2), \delta_{2i}(p_1 p_2 - q_1 q_2)\}, \quad (A\ 9a)$$

a similar result for $B_l$, with $p_n^2 - q_n^2$ and $p_1 p_2 - q_1 q_2$ replaced by $2p_n q_n$ and $p_1 q_2 + p_2 q_1$ respectively, and

$$C_l \equiv C_{ij}^{c,s} = \frac{1}{2}\frac{\lambda}{a_1}(1 + \tfrac{1}{2}k_{ij}^2 a_1^2 - k^2 a_1^2) I_{ij}$$
$$\times \{\delta_{0i}(E_1 + E_2) + \delta_{2i}(E_1 - E_2), \delta_{2i}(p_1 p_2 + q_1 q_2)\}. \quad (A\ 9b)$$

Substituting these results into (3.7) and invoking (1.1$b$) for $\lambda$, (2.7) for $a_{ij}$, (3.8$b, c$) for $E$ and $M$, and summing over $i$ and $j$, we obtain

$$\frac{\lambda}{a_1}(2a_{nml} - \tfrac{1}{2}a_{lmn})\{A_l(p_m p_n - q_m q_n) + B_l(p_m q_n + p_n q_m)\} + \mathscr{E}_l(A_l^2 + B_l^2)$$
$$= -(P_0 + P_2) E^2 + (P_0 - P_2) M^2 \quad (A\ 10a)$$

and
$$\frac{\lambda}{a_1} a_{lmn} C_l(p_m p_n + q_m q_n) - 2C_l C_l = (Q_0 + Q_2) E^2 - Q_2 M^2, \quad (A\ 10b)$$

where
$$P_i = \frac{(3T^2 + 1 - \tfrac{1}{2}\kappa_{ij}^2 - 2\kappa_{ij}^2 \nu_{ij}^2)^2 I_{ij}^2}{4(4\nu_{ij}^2 - 1)} \quad (j \text{ summed}), \quad (A\ 11)$$

$$Q_i = \tfrac{1}{2}(T^2 - 1 + \tfrac{1}{2}\kappa_{ij}^2)^2 I_{ij}^2 \quad (j \text{ summed}), \quad (A\ 12)$$

$$T = \tanh kd, \quad \kappa_{ij} = \frac{k_{ij}}{k}, \quad \nu_{ij} = \frac{\omega_1}{\omega_{ij}} = \left(\frac{k \tanh kd}{k_{ij} \tanh k_{ij} d}\right)^{\frac{1}{2}}. \quad (A\ 13a, b, c)$$

It follows from (A 11) that the width of the resonant neighbourhood in which $P_i$ is large owing to the proximity of $\omega_{ij}$ and $2\omega$ is proportional to $I_{ij}^2$, which (see table 1) is rather small for the 02 and 22 resonances (see §1), but not for the 01 resonance (for which the critical aspect ratio is $d/a = 0.1523$).

Turning to $a_{jlmn}$ ($j, l, m, n = 1$ or 2), as given by (2.9$b$), we infer from (A 1) that

$$D_{lmn} = \tfrac{1}{2}k_n^2 C_{lmn} \quad (l, m = 1 \text{ or } 2), \quad (A\ 14)$$

by virtue of which (2.9$b$) reduces to

$$a_{jlmn} = \tfrac{1}{2}a_1^2 a_i k_i^4 C_{jml} C_{lni} - 2a_1 D_{jlmn} \quad (i \text{ summed};\ j, l, m, n = 1 \text{ or } 2), \quad (A\ 15)$$

The $D_{jlmn}$ $(j, l, m, n = 1 \text{ or } 2)$ are given by (2.5) and (2.10c) according to

$$D_{1111} = D_{2222} = [\psi_1^2(\psi_{1r}^2 + r^{-2}\psi_{1\theta}^2)] \quad \text{(A 16a)}$$

$$= (4a^2N^4)^{-1}\int_0^{ka}\{3J_1^2(x)\,J_1'^2(x)\,x + J_1^4(x)\,x^{-1}\}\,\mathrm{d}x \quad \text{(A 16b)}$$

$$= \tfrac{1}{4}k^2K_1, \quad \text{(A 16c)}$$

$$D_{1122} = D_{2211} = [\psi_1^2(\psi_{2r}^2 + r^{-2}\psi_{2\theta}^2)] \quad \text{(A 17a)}$$

$$= (4a^2N^4)^{-1}\int_0^{ka}\{J_1^2(x)\,J_1'^2(x)\,x + 3J_1^4(x)\,x^{-1}\}\,\mathrm{d}x \quad \text{(A 17b)}$$

$$= \tfrac{1}{12}k^2(K_1 + 8K_{-1}), \quad \text{(A 17c)}$$

$$D_{1212} = D_{1221} = D_{2121} = D_{2112} = [\psi_1\,\psi_2(\psi_{1r}\,\psi_{2r} + r^{-2}\psi_{1\theta}\,\psi_{2\theta}] \quad \text{(A 18a)}$$

$$= (4a^2N^4)^{-1}\int_0^{ka}\{J_1^2(x)\,J_1'^2(x)\,x - J_1^4(x)\,x^{-1}\}\,\mathrm{d}x \quad \text{(A 18b)}$$

$$= \tfrac{1}{12}k^2(K_1 - 4K_{-1}), \quad \text{(A 18c)}$$

wherein integration by parts yields

$$\int_0^{ka} J_1^2(x)\,J_1'^2(x)\,\mathrm{d}x = \frac{1}{3}\int_0^{ka} J_1^4(x)\,(x - x^{-1})\,\mathrm{d}x, \quad \text{(A 19)}$$

$$K_1 \equiv N^{-4}(ka)^{-2}\int_0^{ka} J_1^4(x)\,x\,\mathrm{d}x = 2.3361, \quad K_{-1} \equiv N^{-4}(ka)^{-2}\int_0^{ka} J_1^4(x)\,x^{-1}\,\mathrm{d}x = 1.2433. \quad \text{(A 20a, b)}$$

Combining (A 15)–(A 18) in (3.7), and invoking $\lambda/\alpha_1 = T^2$, we obtain

$$\frac{1}{4}\frac{\lambda^2}{\alpha_1}\{\alpha_{1111}(E_1^2 + E_2^2) + 2\alpha_{1122}(E_1\,E_2 + \tfrac{1}{2}M^2) + 2(\alpha_{1212} + \alpha_{1221})\,(E_1\,E_2 - \tfrac{1}{2}M^2)\}$$
$$= \tfrac{1}{2}(R_0 + R_2 - \tfrac{1}{4}K_1\,T^2)\,E^2 + \tfrac{1}{2}(R_2 - R_0 + \tfrac{1}{3}K_1\,T^2 - \tfrac{4}{3}K_{-1}\,T^2)\,M^2, \quad \text{(A 21)}$$

where
$$R_i = \tfrac{1}{4}\kappa_{ij}^4\,\nu_{ij}^2\,I_{ij}^2 \quad (j \text{ summed}). \quad \text{(A 22)}$$

Substituting (A 10) and (A 21) into (3.7), we obtain (3.11) and (3.12) with

$$A = -P_0 - P_2 + Q_0 + Q_2 + R_0 + R_2 - \tfrac{1}{4}K_1\,T^2, \quad \text{(A 23a)}$$

$$B = P_0 - P_2 - Q_2 - R_0 + R_2 + \tfrac{1}{3}(\tfrac{1}{4}K_1 - 4K_{-1})\,T^2. \quad \text{(A 23b)}$$

The results plotted in figures 1 and 2 are based on single-term $(j = 1)$ approximations to the series $P_0, P_2, \ldots$, which are adequate for an accuracy of better than 0.1 % except in small neighbourhoods (with dimensionless bandwidths of order of $10^{-4}$) of the resonances between the dominant mode and either the 02 or 22 secondary modes. The resonance between the dominant mode and the 01 mode is much stronger, and the corresponding pole in $P_0$ implies $A = B = \infty$ at $d/a = 0.1523$. It is worth noting that this singularity does not affect $A + B$.

## Appendix B. The angular momentum

The angular momentum associated with the free-surface displacement (2.1) is given by

$$\mathscr{M} = \pi a^2 \rho \left[ \int_{-d}^{\eta} \phi_\theta \, \mathrm{d}y \right], \tag{B 1}$$

where $\rho$ is the density and $\phi$, the velocity potential, is given by (I (2.4*b*))

$$\phi = \phi_n(t)\, \psi_n(r,\theta) \operatorname{sech} k_n d \cosh k_n(y+d). \tag{B 2}$$

Substituting (B 2) into (B 1) and expanding in powers of $\eta$, we obtain the first approximation

$$\mathscr{M} = \pi a^2 \rho [\eta \phi_n \psi_{n\theta}]. \tag{B 3}$$

It then suffices, within $1+O(\epsilon^2)$, to approximate $\phi_n$ by $a_n \dot{\eta}_n$ (I (2.11), (2.14)) and to retain only the dominant modes, thereby reducing (B 3) to

$$\mathscr{M} = \pi a^2 a_1 \rho(\eta_1 \dot{\eta}_2 - \eta_2 \dot{\eta}_1) \tag{B 4a}$$

$$= \pi a^2 a_1 \rho \omega_1 \epsilon^2 \lambda^2 (p_1 q_2 - p_2 q_1), \tag{B 4b}$$

where (B 4*b*) follows from (B 4*a*) with the aid of (3.1).

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